A Threshold Signature Primer

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SECTION 1

The Elevator Pitch

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ternary zany canary topiary aviaries

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...how can you cryptographically ensure that no one can spend the funds themselves?

what if you want to just have one signature?

what if you only want majority approval?

SECTION 2

Multi-Signatures & Threshold Signatures

- **Signature:** Alice signed this message.
- **Multi-Signature:** Alice, Bob, and Charlie signed it.
- **Threshold-Signature:** 2 people among Alice, Bob, and Charlie signed it.

“Silent” vs “Loud” multisigs

- **Loud**: We can define a “signature” to consist of several signatures.
- **Silent**: The group produces the same signature as a single signer.

Why “silent”?

Why “silent”?

- Keep a stable identity over time.
- Privacy.
- Interoperability.

Applications

Applications

- Cryptocurrency wallets
- Institutional custody
- Light Client Certificates

Goals

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You will be a **BIG SHOT** at the following topics:

- polynomial secret sharing,
- using the homomorphic properties of group commitments to great profits,
- distributed key generation
- threshold BLS and Schnorr signing.

Warning:

The US Government has declared the contents of this talk to be ~~outdated~~ **illegal** by the year **2031**

SECTION 3

ECC Background

From Elliptic Curves to Cryptographic Groups

- Elliptic curves are a particular model…

From Elliptic Curves to Cryptographic Groups

- Elliptic curves are a particular model…
- of a pretty simple abstraction.

A Group

- $A, B, C, ... \in \mathbb{G}$
- $0 \in \mathbb{G}$
- $A + B$
- $-A$

A Field

- $x, y, z, ... \in \mathbb{F}$
- $0, 1 \in \mathbb{F}$
- $x + y$
- $-x$
- $x \cdot y$
- $\frac{x}{y}$

Addition Rules

For $\mathbb{G}$ and $\mathbb{F}$:

- $$(a + b) + c = a + (b + c)$$
- $$a + b = b + a$$
- $$a + 0 = a$$
- $$a - a := a + (-a) = 0$$

Multiplication Rules

For $\mathbb{F}$:

- $x(yz) = (xy)z$
- $xy = yx$
- $x1 = x$
- $\frac{x}{x} = 1$ (except 0)
- $x(y + z) = xy + xz$

The Field acts on the Group

For $x, y \in \mathbb{F}$, and $A, B \in \mathbb{G}$:

- $$x \cdot A$$
- $$(x + y) \cdot A = x \cdot A + y \cdot A$$
- $$xyA = x(yA)$$

Hard Problems

We pick a point $G \in \mathbb{G}$, and then:

- Given $X := x \cdot G$, finding x is hard.
  (Discrete Logarithm)
- Given $X := x \cdot G, Y := y \cdot G$, finding $Z := (xy) \cdot G$ is hard.
  (Computational Diffie Hellman).
- Given X, Y, distinguishing $xy \cdot G$ from a random $P \in G$ is hard.
  (Decisional Diffie Hellman).

SECTION 4

Signatures

$$\mathrm{Gen}() : (\mathrm{Sk},\ \mathrm{Pk})$$

$$\mathrm{Sign}() : (\mathrm{Pk}, \mathrm{Sk}, M) \to \mathrm{Sig}$$

$$\mathrm{Verify}() : (\mathrm{Pk}, M, \mathrm{Sig})$$

$$\mathrm{Gen}() : (\mathrm{Sk},\ \mathrm{Pk})$$

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$$\mathrm{Verify}() : (\mathrm{Pk}, M, \mathrm{Sig})$$

Security:

- you can sign $\sim$ you know sk,
- (can't change what message a signature is about)

SECTION 5

Schnorr Signatures

Algorithm

$$\underline{\text{Gen}()}$$

$$x \xleftarrow{\$} \mathbb{F}$$

$$(x, x \cdot G)$$

$$\underline{\text{Sign(X, m)}}$$

$$k \xleftarrow{\$} \mathbb{F}$$

$$K := k \cdot G$$

$$c := H(X, m)$$

$$s := k + cx$$

$$(K, s)$$

$$\underline{\text{Verify(X, m, (K, s))}}$$

$$s \cdot G \stackrel{?}{=} K + c \cdot X$$

Correctness

$$s \cdot G = (k + cx) \cdot G = k \cdot G + (cx) \cdot G = K + c \cdot X$$

SECTION 6

Pairings

From One Group to Three

$$\mathbb{G} \longrightarrow \mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_T$$

From One Group to Three

$$\mathbb{G} \longrightarrow \mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_T$$

but still sharing $\mathbb{F}$.

A New Operation

$$\odot : (\mathbb{G}_1, \mathbb{G}_2) \to \mathbb{G}_T$$

Rules

- $$(A + B) \odot C = A \odot C + B \odot C$$
- $$(xA) \odot B = x(A \odot B) = A \odot (xB)$$

MULTIPLICATIVE NOTATION (+ e)

= LIE

group law:	gh	← looks like field mult
inverse:	g^{-1}	← why not $1/g$???
identity:	1	← the number 1 ? really ??
exponentiation:	g^x	← confusing... is g to the power x a field thing? ugh
scalar mult:	g^x	←
addition:	gh	← looks like field mult again
scalar addition:	$g^x h^y$	← what is this ??
pairing:	$e(g, h)$	← e what ??

looks like field stuff

people forget this isn't commutative mult

ugly and inconsistent

harder to read and teach

VS

ODOT NOTATION

= OBVIOUSLY SUPERIOR

group law:	$G + H$	← like vector addition (makes sense)
inverse:	$-G$	← clear and clean
identity:	0	← 0
exponentiation:	$x\,G$	← obviously a scalar times a point
scalar mult:	$x\,G$	← consistent and clear
addition:	$G + H$	← consistent and clear
scalar addition:	$x\,G + y\,H$	← perfect
pairing:	$G \odot H$	← clean and unambiguous

matches geometry and linear algebra

consistent and easy to learn

no confusion with fields

clearly shows what's happening (points + scalars)

Multiplicative notation: looks like a lie, teaches bad habits, causes pain

$\neq$

Odot notation: honest, natural, beautiful, and mathematically sound ♡

Aside: Performance

Approximately:

- $\mathbb{F}$ operations: cheap
- $\mathbb{G}$ operations: 10x more expensive
- $\mathbb{F} \cdot \mathbb{G}$: 10x again
- $\odot$: another 10x

In summary: minimizing pairings is very impactful to performance.

Note: $\sum_i A_i \odot B_i$ is faster than n pairings.

SECTION 7

BLS Signatures

The Algorithm

$$\underline{\mathrm{Gen}()}$$

$$x \xleftarrow{\$} \mathbb{F}$$

$$(x, x \cdot G_1)$$

$$\underline{\mathrm{Sign(m)}}$$

$$xH(m)$$

$$\underline{\mathrm{Verify(X,\ m,\ \sigma)}}$$

$$G_1 \odot \sigma \stackrel{?}{=} X \odot H(m)$$

Correctness

$$G_1 \odot \sigma = G_1 \odot (xH(m)) = (xG_1) \odot H(m) = X \odot H(m)$$

Pitfall

$$H'(m) := \mathbb{F}(H(m)) \cdot G_2$$

Is this secure?

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Is this secure?

$$X \odot H'(M) = X \odot (\mathbb{F}(H(m)) \cdot G_2) = (\mathbb{F}(H(m))X) \odot G_2$$

SECTION 8

BLS Multi-Signatures

SECTION 9

$$X_1 \odot H(m) \stackrel{?}{=} G_1 \odot \sigma_1$$

$$X_2 \odot H(m) \stackrel{?}{=} G_1 \odot \sigma_2$$

what if we just did...

$$X_1 \odot H(m) \stackrel{?}{=} G_1 \odot \sigma_1$$

$$X_2 \odot H(m) \stackrel{?}{=} G_1 \odot \sigma_2$$

what if we just did...

$$X_1 \odot H(m) + X_2 \odot H(m) \stackrel{?}{=} G_1 \odot \sigma_1 + G_1 \odot \sigma_2$$

$$X_1 \odot H(m) \stackrel{?}{=} G_1 \odot \sigma_1$$

$$X_2 \odot H(m) \stackrel{?}{=} G_1 \odot \sigma_2$$

what if we just did...

$$X_1 \odot H(m) + X_2 \odot H(m) \stackrel{?}{=} G_1 \odot \sigma_1 + G_1 \odot \sigma_2$$

$$(X_1 + X_2) \odot H(m) \stackrel{?}{=} G_1 \odot (\sigma_1 + \sigma_2)$$

Is this secure?

Kinda

Here's one attack:

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$$A \odot H(m), B \odot H(m)$$

Kinda

Here's one attack:

$$A \odot H(m), B \odot H(m)$$

$$C = -(A + B)\ (A + B - (A + B)) \odot H(m) = 0 \odot H(m) = 0$$

so

$\sigma = 0$ works

Why Kinda?

The attacker doesn't know x such that: $x \cdot G_1 = -(A + B)$. By requiring this knowledge (e.g. with a signature), we thwart this attack.

The Ultimate Trick to Avoid These Issues

$$r_i \xleftarrow{\$} \mathbb{F}$$

$$(r_1 X_1 + ... + r_n X_n) \odot H(m) = G_1 \odot (r_1 \sigma_1 + ...)$$

The Ultimate Trick to Avoid These Issues

$$r_i \xleftarrow{\$} \mathbb{F}$$

$$(r_1 X_1 + ... + r_n X_n) \odot H(m) = G_1 \odot (r_1 \sigma_1 + ...)$$

Or, in general:

$$\forall i, \varphi(a_i, b_i, ...) = 0$$

$$\sum_i r_i \varphi(a_i, b_i, ...) = 0$$

Batch Verification

$$\forall i, X_i \odot H(m_i) = G_1 \odot \sigma_i$$

$$\forall i, X_i \odot H(m_i) - G_1 \odot \sigma_i$$

$$\sum_i (r_i X_i \odot H(m_i)) - G_1 \odot \left(\sum_i \sigma_i \right)$$

SECTION 10

Secret Sharing

Multi-Sigs are Linear

$$x = x_1 + x_2 + ... + x_n$$

need all the keys.

What if we only want to need t of them?

The solution to 82.7% of math problems?

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Polynomials.

Shamir's Secret Sharing

Given t coefficients:

$$f(X) := a_0 + a_1 X + ... + a_{\{t-1\}} X^{\{t-1\}}$$

any t points:

$$f(p_0), f(p_1), ..., f\left(p_{\{t-1\}}\right)$$

determine f exactly

(Intuition: $f(p) = 0 \Leftrightarrow f = (X - p)...$)

The Sharing

$$x := f(0)$$

$$x_i := f(p_i)$$

can locally compute λ_i such that:

$$\sum_i \lambda_i x_i = f(0)$$

Some Useful Properties

$$f(x) + g(x) = (f + g)(x) = (f_0 + g_0) + (f_1 + g_1)x + ...$$

$$F := f \cdot G$$

$$F(x) = (a_0 G) + x(a_1 G) + ... = f(x) \cdot G$$

Local operations on secret shares

$$f(p_i) + g(p_i) = (f + g)(p_i)$$

$$\alpha f(p_i) = (\alpha f)(p_i)$$

Galaxy Brained Alternative

For each subset S of t signers, generate a linear sharing

$$x = x_{S1} + x_{S2} + ...$$

When a particular subset S wants to sign, use x_{Si} as your share.

What if you don't know what subset will be present?

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You do need an exponential number of shares...

but that didn't stop Monero from deploying this for their wallet.

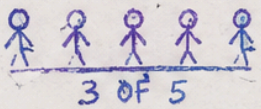
SPAMTON G. SPAMTON ON SECRET SHARING

SHAMIR'S SECRET SHARING?

WHAT A SCAM!!!

It's too simple!!! It doesn't serve my [$!@#]ing AMBITION!!!

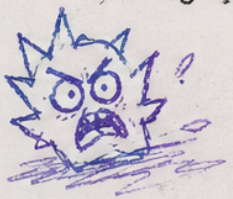
- FIXED THRESHOLD?? So rigid!!! what if i need 3 today but 5 tomorrow!?



NOT FLEXIBLE! :(

- CAN'T PICK WHO CAN COMBINE??? what if i don't trust someone today!?

- NO SUBSET MAGIC!!! what if i want ANY group to be able to sign!?



SO LIMITING...

MORE SHARES!
MORE OPTIONS!
MORE [[KROMER]]!!!

INSTEAD, DO THIS:

LINEAR SECRET SHARES FOR EVERY POSSIBLE SUBSET!!!

ANY GROUP!
ANY SIZE!
ANY TIME!
PERFECT!!!

FOR ALL $S \subseteq \{1...n\}$, $|S| \geq t$
PRECOMPUTE LINEAR SHARES!

THERE ARE $2^n - 1$ SUBSETS!
WE COVER THEM ALL!!!

STORE YOUR EXPONENTIALLY SIZED SHARE SET WITH ME!
$$$$$$$/MONTH
PREMIUM KEY STORAGE
(WORTH EVERY [KROMER])

- REDUNDANCY? THAT'S CALLED FLEXIBILITY!!!
- STORAGE? THAT'S CALLED INVESTING!!! :)

SECTION 11

BLS Threshold Signatures

$$x \cdot H(m) = f(0) \cdot H(m) = \lambda_i x_i \cdot H(m) = \lambda_i \cdot (x_i \cdot H(m)) = \lambda_i \cdot \sigma_i$$

SECTION 12

Distributed Key Generation

Warmup: Linear Sharing

$$x = x_1 + x_2 + ... + x_n$$

$$X = x \cdot G$$

Protocol

How can each person get their linear share x_i?

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Now, how can we learn $\sum_i x_i \cdot G$?

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$$x_i \xleftarrow{\$} \mathbb{F}$$

Now, how can we learn $\sum_i x_i \cdot G$?

$$X_i := x_i \cdot G \rightrightarrows$$

$$X := \sum_i X_i$$

Trusted Key Generation

$$f \xleftarrow{\$} \mathbb{F}[X]_{\leq t}$$

$$f(p_i) \longrightarrow P_i$$

$$F := f \cdot G$$

$$F$$

$F(0)$ is the public key, and $F(p_i)$ lets us verify partial signatures.

Honest Distributed Key Generation

$\underline{\text{Dealer}()_i}$

$$f_i \xleftarrow{\$} \mathbb{F}[X]_{\leq t}$$

$$f_i(p_j) \longrightarrow P_j$$

$$F_i := f_i \cdot G$$

$$F_i \rightrightarrows$$

$\underline{\text{Player}()_j}$

$$D_i \longrightarrow f_i(p_j)$$

$$f(p_j) := \sum_i f_{i(p_j)}$$

$$\rightrightarrows F_i$$

$$F := \sum_i F_i$$

$$\left(F, f(p_j)\right)$$

Key Insight

$$\sum_i f_i(p_j) = \left(\sum_i f_i\right)(p_j)$$

Problem Points

- What if I send $s_{ij} \neq f_i(p_j)$?
- What if I send different F_i to different players?
- What if I don't send anything at all?

SECTION 13

Two Round DKG

The Idea

- $f_i(p_j) \stackrel{?}{=} F_i(p_j)$ lets us check a dealer's share.
- We can complain if it's wrong.
- We can exclude bad dealers.
- Dealers reveal shares to protect against bad players.

The Protocol

$$\underline{\text{Dealer}()_i}$$

$$f_i \xleftarrow{\$} \mathbb{F}[X]_{\leq t}$$

$$F_i := f_i \cdot G$$

$$F_i \rightrightarrows$$

$$s_{ij} := f_i(p_j) \longrightarrow P_j$$

$$\sigma_j \longleftarrow ? \; P_j$$

$$\sigma_j \text{ or } s_{ij} \rightrightarrows$$

$$\underline{\text{Player}()_j}$$

$$\rightrightarrows F_i, D_i \longrightarrow s_{ij}$$

$$f(p_j) := \sum_i f_{i(p_j)}$$

$$\text{if } F_i(p_j) \stackrel{?}{=} s_{ij}$$

$$D_i \longleftarrow \text{sign}(D_i, F_i, \ldots)$$

$$\text{good} := \{D_i \mid \sigma_i \text{ good} \vee s_{ij} \text{ good}\}$$

$$F := \sum_{i \in \text{ good}} F_i, s_j := \sum_{i \in \text{ good}} s_{ij}$$

$$(F, s_j)$$

Questions to ponder

- How many dealers need to be good to get an acceptable result?

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- What needs to be signed by the player?
- Is it necessary to broadcast F_i?

Questions to ponder

- How many dealers need to be good to get an acceptable result?
- What should a dealer do if they have $\geq t$ reveals?
- What if too many dealers reveal the share for a given player?
- What needs to be signed by the player?
- Is it necessary to broadcast F_i?
- What happens if players disagree about the dealer's final message?

SECTION 14

Practical Challenges

Synchrony

- How long should I wait to receive an acknowledgement?

Synchrony

- How long should I wait to receive an acknowledgement?
- How long should I wait for a dealer to reveal?

Observability

- How can external observers be convinced the protocol ran correctly?

Faulty Assumptions

- The protocol ties its security to assumptions about faults.

SECTION 15

One Round DKG

Idea

- private share $\Rightarrow$ public ciphertext
- complaints $\Rightarrow$ ZK Proof

Initial Protocol

$$f_i \xleftarrow{\$} \mathbb{F}[X]_{\leq t}$$

$$F_i := f_i \cdot G$$

$$c_{ij} := \mathrm{Enc}\big(\mathrm{sk}_i, \mathrm{pk}_j, f_i(p_j)\big)$$

$$\pi_{ij} := \mathrm{Prove}\big(\mathrm{sk}_i, s_{ij} \mid c_{ij} = \mathrm{Enc}\big(\mathrm{sk}_i, \mathrm{pk}_j, s_{ij}\big) \wedge s_{ij} \cdot G = S_{ij}\big)$$

Dealer posts c_{ij}, π_{ij}, S_{ij}, and everyone immediately knows whether the dealer is honest, and has their share.

Improving on this

What operations does Enc do?

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- $$P := \mathrm{sk}_i \cdot \mathrm{pk}_j$$

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- $\mathrm{AES}\big(K, s_{ij}\big)$?

Improving on this

What operations does Enc do?

- $P := \mathrm{sk}_i \cdot \mathrm{pk}_j$
- $K := H(P)$
- $\mathrm{AES}\big(K, s_{ij}\big)$?
- instead, just do $K \in \mathbb{F}$, and do $K + s_{ij}$

Draft Two

$$\mathrm{Exch} : (\mathrm{Sk}, \mathrm{Pk}) \to \mathbb{F}$$

$$f_i \xleftarrow{\$} \mathbb{F}[X]_{\leq t}$$

$$F_i := f_i \cdot G$$

$$m_{ij} := \mathrm{Exch}\big(\mathrm{sk}_i, \mathrm{pk}_j\big)$$

$$M_{ij} := m_{ij} \cdot G$$

$$c_{ij} := m_{ij} + f_i\big(p_j\big)$$

$$\pi_{ij} := \mathrm{Prove}\big(\mathrm{sk}_i, \mid m_{ij} = \mathrm{Exch}\big(\mathrm{sk}_i, \mathrm{pk}_j\big) \wedge m_{ij} \cdot G = M_{ij}\big)$$

then broadcast M_{ij}, π_{ij}, c_{ij}

Players check $c_{ij} \cdot G \stackrel{?}{=} M_{ij} + F_i\big(p_j\big)$

One Trick

In some proof systems (Bulletproofs), something like:

$$\{x_1, ... \mid \text{stuff}(x_1, ...) \wedge \text{x\_i dot.c G} = \text{X\_i}\}$$

is partially free.

Intuition: ZK Proof = Checks + Commitment Scheme

$x_i \cdot G$ commits to x_i.

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Intuition: ZK Proof = Checks + Commitment Scheme

$x_i \cdot G$ commits to x_i.

Hence:

$$\text{Prove}\big(\text{sk}_i \mid m_{ij} = \text{Exch}\big(\text{sk}_i, \text{pk}_j\big)\big)$$

Another Trick

In many schemes (again Bulletproofs), proof size << statement size, e.g. $O(n \lg n)$

If we have n proofs, maybe...

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If we have n proofs, maybe...

prove them all together, rather than one at a time.

We can also verify n proofs much faster than one proof n times.

A Final Trick

We have $\mathbb{G}$ with scalars $\mathbb{F}$.

Our computation is in $\mathbb{F}$.

We want $\mathrm{Exch}\big(\mathrm{sk}_i, \mathrm{pk}_j\big)$. What group to use?

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We have $\mathbb{G}$ with scalars $\mathbb{F}$.

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- We want a group $\mathbb{H}$ where operations are arithmetic over $\mathbb{F}$.

A Final Trick

We have $\mathbb{G}$ with scalars $\mathbb{F}$.

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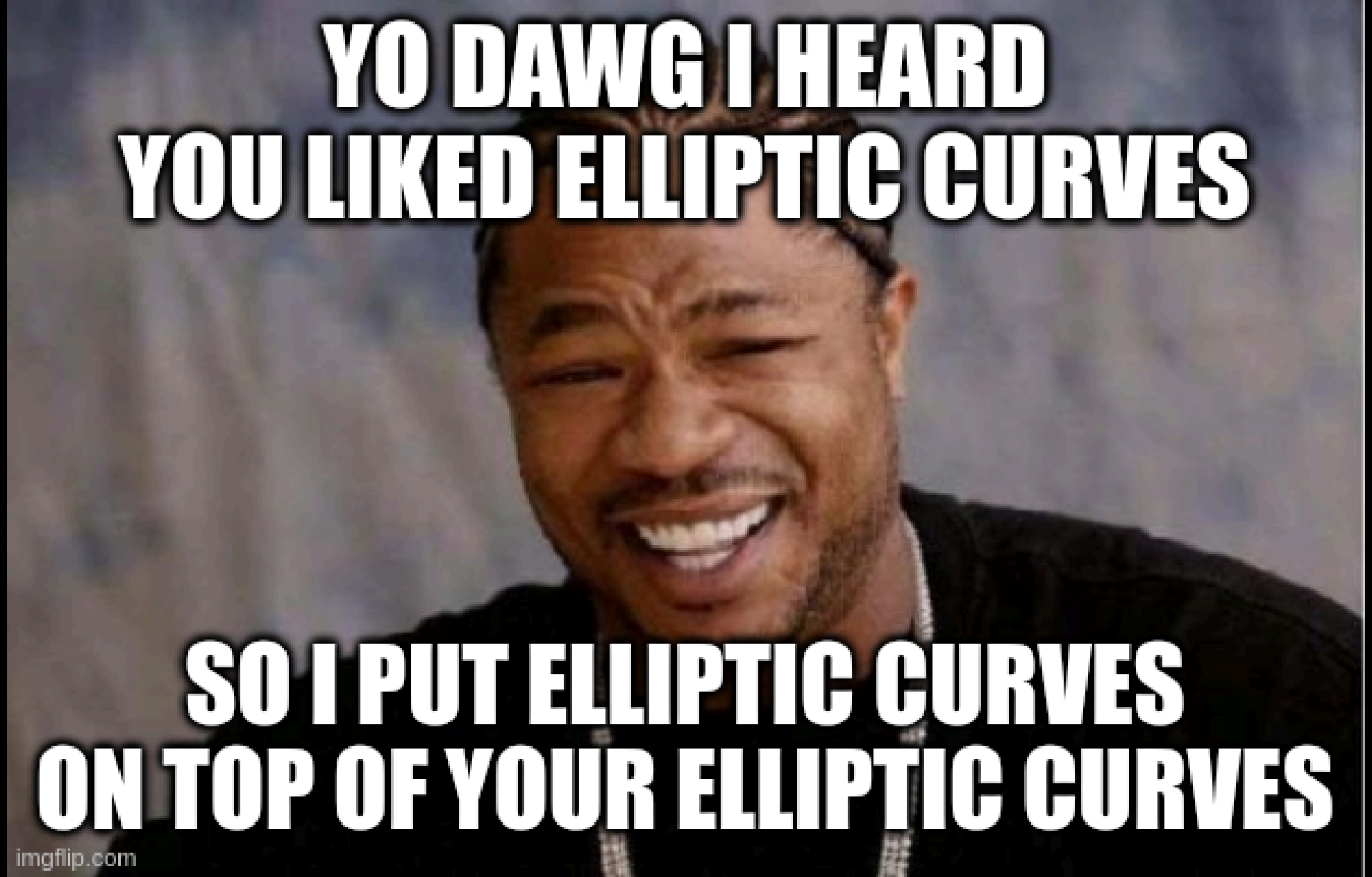
We want $\text{Exch}\left(\text{sk}_i, \text{pk}_j\right)$. What group to use?

- We want a group $\mathbb{H}$ where operations are arithmetic over $\mathbb{F}$.
- An Elliptic Curve with coordinates in $\mathbb{F}$ can be found.


YO DAWG I HEARD
YOU LIKED ELLIPTIC CURVES
SO I PUT ELLIPTIC CURVES
ON TOP OF YOUR ELLIPTIC CURVES
imgflip.com

Examples of this trick

- Bandersnatch

Examples of this trick

- Bandersnatch
- Balthizard

Examples of this trick

- Bandersnatch
- Balthizard
- Baby Jub Jub

Examples of this trick

- Bandersnatch
- Balthizard
- Baby Jub Jub
- Werewerewire

Examples of this trick

- Bandersnatch
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- Organikk

Examples of this trick

- Bandersnatch
- Balthizard
- Baby Jub Jub
- Werewerewire
- Organikk
- Pasta curves

Should we always use the one-round DKG?

Two Example Environments

Institutional Custody

- Offline signing, communication over group chat,
- air-gapped secret-share storage,
- completely private operation,
- DKG happens once.

Validator Certificate Key

- DKG happens often,
- completely online and connected,
- convenient public blockchain to post information.

Some cons of the one-round DKG

- Latency might be a lot worse (minutes of computation for 100 people).
- Bugs in ZK circuit.
- Implementing broadcast might take two rounds anyways.

Reshare

What if we want to keep the public key X, but change the shares?

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$$? \sum_{i \in \text{ good}} f_i ?$$

$$\left(\sum_{i \in \text{ good}} \lambda_i f_i \right)(0) = \sum_{i \in \text{ good}} \lambda_i x_i = x$$

SECTION 16

Schnorr Threshold Signatures

Recap

$$
\begin{aligned}
k &\xleftarrow{\$} \mathbb{F} \\
K &:= k \cdot G \\
c &:= H(X, K, m) \\
s &:= \mathrm{k} + c\ \mathrm{x}
\end{aligned}
$$

What tools can we use to do this in a threshold setting?

Recap

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What tools can we use to do this in a threshold setting?

How to do $\mathrm{k} \xleftarrow{\$} \mathbb{F}$ jointly?

How to do $\mathrm{k} + c\ \mathrm{x}$ jointly?

“Lazy” Threshold Schnorr

$$(X, x_i) := \mathrm{DKG}()$$

———

$$(K, k_i) := \mathrm{DKG}()$$

$$c := H(X(0), K(0), m)$$

$$s_i := k_i + cx_i$$

$$s_i \rightrightarrows$$

$$s_i \stackrel{?}{=} K(p_i) + cX(p_i)$$

$$s := \sum_i s_i$$

$$(K, s)$$

DKG Component Tweaks

DKG Component Tweaks

- Should we use the two-round or one-round DKG?

DKG Component Tweaks

- Should we use the two-round or one-round DKG?
- What if we use a linear secret sharing DKG?

Inlined Two-Round DKG

Attempt:

- Do a DKG for k, and a reshare for x,
- You receive x_{ij} and k_{ij}, compute s_{ij}, send that.
- Use the complaints to compute:

$$K := \sum_{j \in \text{ good}} \lambda_j K_j(0)$$

$$s := \sum_{i,j \in \text{ good}} \lambda_i \lambda_j s_{ij}$$

Does this work?

SECTION 17

Threshold ECDSA

The Annoying Algorithm

$$k \xleftarrow{\$} \mathbb{F}$$

$$R := \left(\frac{1}{k}\right) G$$

$$s := k(m + h(R)x)$$

$$(R, s)$$

tricky part:

$$s = \mathrm{k}\ m + h(R)\ \mathrm{k}\ \mathrm{x}$$

Multiplication is hard, so do it in advance

If a trusted third party gives us $a(p_j), b(p_j), c(p_j) s.t. a(0)b(0) = c(0)$, can we use that?

Multiplication is hard, so do it in advance

If a trusted third party gives us $a(p_j), b(p_j), c(p_j) s.t. a(0)b(0) = c(0)$, can we use that?

$$(k + a)x - a(x + b) + c$$

$$kx + ax - ax - ab + c$$

$$kx$$

if we use a and b just once, then $k + a$ hides k, and $x + b$ hides b.

Idea: compute $k + a$, $x + b$ secret shared, then "open" the values. With $k + a$ and $x + b$ public, we can compute kx secret shared.

Inversion?

We also need to compute $\left(\frac{1}{k}\right)G$.

If we generate a mask $d \xleftarrow{\$} \mathbb{F}$, then we can safely reveal kd, and dG

Inversion?

We also need to compute $\left(\frac{1}{k}\right)G$.

If we generate a mask $d \xleftarrow{\$} \mathbb{F}$, then we can safely reveal kd, and dG

then, make kd public, letting us compute

$$\left(\frac{1}{k}d\right)(dG) = \left(\frac{1}{k}\right)G$$

Methods to compute Beaver Triples (the hard part)

- Paillier Encryption and fancy ZK Proofs

  (bad)

- Oblivious Transfer

  (also bad, unfortunately)

ECDSA

- ✗ SLOW
- ✗ COMPLICATED
- ✗ NOT LINEAR
- ✗ WEIRD NONCE (uh oh bad nonce)
- ✗ HARD TO COMBINE

SCHNORR

- ✓ FAST
- ✓ SIMPLE
- ✓ LINEAR
- ✓ SAFE NONCES
- ✓ EASY TO COMBINE

HERE'S A KROMER KID, $
NOW GET YOURSELF A
REAL SIGNATURE SCHEME!!!

SECTION 18

Some Questions to Ponder

- Do we need the message before starting threshold signing?
- Can we amortize signing multiple things at once?
- Can we pipeline these protocols?
- Do post-quantum schemes work this nicely?

Thank You!